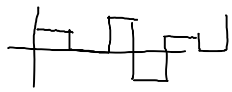
UNITEC

25 Marzo 2011

UNIDAD 7 Transformada de laplace

$$y'' + a_1 y' + a_2 y = f(t)$$

$f(t)$ no es continua, es una función escalonada



$f(t)$

Laplaciano $\mathcal{L}$

Teorema Fundamental

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)\, e^{-st}\, dt = F(s)$$

1

UNITEC

FORMULARIO

TRANSFORMADA

INVERSA

① $\mathcal{L}\{c\} = \frac{c}{s}$ $\qquad$ $\mathcal{L}^{-1}\left\{\frac{c}{s}\right\} = c$

② $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$ $\qquad$ — — —— — — —

③ $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

④ $\mathcal{L}\{\text{Sen } wt\} = \frac{w}{s^2+w^2}$

⑤ $\mathcal{L}\{\text{Cos } wt\} = \frac{s}{s^2+w^2}$

⑥ $\mathcal{L}\{\text{Senh } wt\} = \frac{w}{s^2-w^2}$

⑦ $\mathcal{L}\{\text{Cosh } wt\} = \frac{s}{s^2-w^2}$

2

UNITEC
Universidad Tecnológica de México

Ejercicio Hallar la transformada de:

a) $f(t) = (t^2-5)^2 = t^4 - 10t^2 + 25$

$\mathcal{L}\{t^4\} - \mathcal{L}\{10t^2\} + \mathcal{L}\{25\}$

$\frac{4!}{s^5} - 10\left[\frac{2!}{s^3}\right] + \frac{25}{s} = \frac{24}{s^5} - \frac{20}{s^3} + \frac{25}{s}$

b) $f(t) = 5e^{4t} - 3t^6 + 4\,\mathrm{Sen}\,3t - 2\cos 2t$

$\mathcal{L}\{f(t)\} = 5\left[\frac{1}{s-4}\right] - 3\left[\frac{6!}{s^7}\right] + 4\left[\frac{3}{s^2+9}\right] - 2\left[\frac{s}{s^2+4}\right]$

$F(s) = \frac{5}{s-4} - \frac{2160}{s^7} + \frac{12}{s^2+9} - \frac{2s}{s^2+4}.$

3

UNITEC
Universidad Tecnológica de México

c) $f(t) = \boxed{t^4}\boxed{e^{-3t}}$

Teorema Traslación

① Identificar $f(t)$ que multiplica e^{at}

$\mathcal{L}\{f(t)\cdot e^{at}\}$

② Aplicar $\mathcal{L}\{f(t)\} = F(s)$

③ Identificar de e^{at} el valor de a

④ $F(s)\Big|_{s-a} = F(s-a)$

$f(t) = t^4$

$\mathcal{L}\{f(t)\} = \frac{4!}{s^5} = F(s)$

$a = -3$

$F(s)\Big|_{s-a} = F(s-a)$

$\boxed{F(s-a) = \frac{4!}{(s+3)^5}}$

4

UNITEC
Universidad Tecnológica de México

d) $f(t) = \underbrace{e^{2t}}\underbrace{\text{Sen}\,5t}_{f(t)}$ $\quad F(s) = \mathcal{L}\{f(t)\} = \dfrac{5}{s^2+25}$

$a = 2$

$$F(s-a) = \frac{5}{(s-2)^2+25} = \frac{5}{s^2-4s+4+25} = \boxed{\frac{5}{s^2-4s+29}}$$

e) $f(t) = t \quad \cos 5t$ $\Big\}$ Método de la derivada.

$\mathcal{L}\{t^n \cdot f(t)\}$

$t^n = t \quad \boxed{n=1}$ Veces que se deriva

$f(t) = \cos 5t \xrightarrow{\mathcal{L}} F(s) = \dfrac{s}{s^2+25}$

$$\boxed{\mathcal{L}\{t^n \cdot f(t)\} = (-1)^n \left[d^n F(s) \right]}$$

5

UNITEC
Universidad Tecnológica de México

$F(s) = \dfrac{s}{s^2+25} \Rightarrow$ derivar $\qquad \dfrac{u}{v} = \dfrac{vu' - uv'}{v^2}$

$$F'(s) = \frac{(s^2+25)(1) - (s)(2s)}{(s^2+25)^2} = \frac{s^2+25-2s^2}{(s^2+25)^2}$$

$$(-1) \quad \frac{-s^2+25}{(s^2+25)^2} \Rightarrow \boxed{\frac{s^2-25}{(s^2+25)^2}}$$

f) $f(t) = \underbrace{\cos\left(t - \frac{\pi}{2}\right)}\underbrace{u\left(t-\frac{\pi}{2}\right)}_{\text{Escalón}}$ $\qquad$ Escalón Unitario $u(t-a)$

$a = \frac{\pi}{2}$

Lo mismo

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as} F(s)$$

6

UNITEC
Universidad Tecnológica de México

$$f(t) = \cos t \xrightarrow{\mathcal{L}} F(s) = \frac{s}{s^2+1}$$

$$\mathcal{L}\left\{\cos\left(t-\frac{\pi}{2}\right)u\left(t-\frac{\pi}{2}\right)\right\} = \boxed{e^{-\frac{\pi}{2}s} \cdot \left[\frac{s}{s^2+1}\right]}$$

g) $f(t) = (6t+5)\,u(t-3)$

$$\boxed{\mathcal{L}\{g(t)\,u(t-a)\} = e^{-as}\,\mathcal{L}\{g(t+a)\}}$$

① $g(t) = 6t+5$

$a = 3$

$$g(t+a) = g(t)\Big|_{t+a} = g(t+3) = 6(t+3)+5 = 6t+23$$

$$\mathcal{L}\{6t+23\} = \frac{6}{s^2} + \frac{23}{s}$$

$$\mathcal{L}\{f(t)\} = \boxed{e^{-3s}\left[\frac{6}{s^2} + \frac{23}{s}\right]}$$

7

UNITEC
Universidad Tecnológica de México

EJERCICIOS

① $f(t) = (\cos t + \operatorname{Sen} 2t)^2$

② $f(t) = (t-5)^3\, e^{-3t}$

③ $f(t) = t^2 \operatorname{Sen} 3t$

29 Marzo 2011, 01 Abril } UNIDAD 8
05 ", 08 " } UNIDAD 9
12 ", 15 " } REPASO

(*) 15 "
26 EXAMEN
29 REVISION

8

UNITEC

UNIDAD # 8.

Transformada Inversa

CONVOLUCIÓN · $f * g$

$$f * g = \int_0^t f(\beta)\, g(t-\beta)\, d\beta$$

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = F(s)G(s)$$

$$\mathcal{L}^{-1}\{F(s)\, G(s)\} = f * g$$

9

UNITEC

EJERCICIO Dados $f(t) = \frac{1}{30} e^{3t}$, $g(t) = \frac{1}{6} e^{-3t}$

a) Convolución

b) $\mathcal{L}\{f * g\}$

Sol.

$$f * g = \int_0^t f(\beta)\, g(t-\beta)\, d\beta$$

$$f(\beta) = \frac{1}{30} e^{3\beta} \qquad g(t-\beta) = \frac{1}{6} e^{-3(t-\beta)}$$

$$f(\beta)\, g(t-\beta) = \frac{1}{30} e^{3\beta} \cdot \frac{1}{6} e^{-3t+3\beta} = \frac{1}{180} e^{3\beta} e^{-3t} e^{3\beta}$$

$$f * g = \frac{1}{180} e^{-3t} \int_0^t e^{6\beta}\, d\beta = \frac{1}{180} e^{-3t} \left[\frac{1}{6} e^{6\beta} \right|_0^t$$

10

UNITEC
Universidad Tecnológica de México

$$f*g = \frac{1}{180} e^{-3t}\left[\frac{1}{6} e^{6t} - \frac{1}{6}(1)\right] = \frac{e^{-3t}}{1080}\left[e^{6t} - 1\right]$$

$$f*g = \frac{e^{3t} - e^{-3t}}{1080} = \boxed{\frac{\operatorname{senh}(3t)}{540}}$$

$$\frac{1}{540} \cdot \boxed{\frac{e^{3t} - e^{-3t}}{2}} = \operatorname{Senh}(3t)$$

b) $\mathcal{L}\{f*g\} = \mathcal{L}\{f(t)\} \cdot \mathcal{L}\{g(t)\}$

$$= \underbrace{\frac{1}{30}\left[\frac{1}{s-3}\right]}_{F(s)} \underbrace{\frac{1}{6}\left[\frac{1}{s+3}\right]}_{G(s)} = \frac{1}{180(s^2-9)}$$

11

UNITEC
Universidad Tecnológica de México

2 Aplicar Convolución para calcular la transformada inversa de laplace de la función

$$H(s) = \frac{1}{(s+1)(s^2-1)} = \underbrace{\frac{1}{s+1}}_{F(s)} \cdot \underbrace{\frac{1}{s^2-1}}_{G(s)}$$

$$\mathcal{L}^{-1}\{F(s)\} = f(t) = e^{t}$$

$$\mathcal{L}^{-1}\{G(s)\} = g(t) = \operatorname{senh} t = \frac{e^{t} - e^{-t}}{2}$$

$$f*g = \int_0^t \underbrace{e^{\beta}}_{f(\beta)} \underbrace{\left[\frac{e^{t-\beta} - e^{-(t-\beta)}}{2}\right]}_{g(t-\beta)} d\beta$$

$$\frac{1}{2}\int_0^t \left[e^{\beta} e^{t} e^{-\beta} - e^{\beta} e^{-t} e^{\beta}\right] d\beta$$

12

UNITEC
Universidad Tecnológica de México

$$f*g = \frac{1}{2}\int_0^t (e^t - e^{-t}e^{2\beta})\,d\beta$$

$$= \frac{1}{2}e^t\int_0^t d\beta - \frac{1}{2}e^{-t}\int_0^t e^{2\beta}\,d\beta$$

$$= \frac{1}{2}e^t\left(\beta\Big|_0^t - \frac{1}{2}e^{-t}\left(\frac{1}{2}e^{2\beta}\Big|_0^t\right.\right.$$

$$= \frac{1}{2}e^t(t) \quad - \frac{1}{4}e^{-t}\left[e^{2t}-1\right]$$

$$\boxed{f*g = \frac{1}{2}te^t - \frac{1}{4}e^t + \frac{1}{4}e^{-t}}$$

13

UNITEC
Universidad Tecnológica de México

Transformada de una derivada

$$\mathcal{L}\{y'''\} = s^3 y(s) - s^2 y(0) - s y'(0) - s^0 y''(0)$$

$$\mathcal{L}\{y''\} = s^2 y(s) - s\, y(0) - s^0 y'(0)$$

$$\mathcal{L}\{y'\} = s\, y(s) - s^0 y(0)$$

$$\mathcal{L}\{y\} = s^0 y(s)$$

14

UNITEC
Universidad Tecnológica de México

EJERCICIO Resolver Condición $y(0) = 3$

$y' - y = 2e^{4t}$ Aplicando Laplace.

① Aplicar $\mathcal{L}$ a toda la ecuación

$$\mathcal{L}\{y'\} - \mathcal{L}\{y\} = 2\,\mathcal{L}\{e^{4t}\}$$

$$sY(s) - y(0) - Y(s) = 2\left(\frac{1}{s-4}\right)$$

② Aplicar condiciones y despejar $Y(s)$

$$sY(s) - 3 - Y(s) = \frac{2}{s-4}$$

$$Y(s)\left[s-1\right] = \frac{2}{s-4} + 3$$

$$Y(s) = \frac{2}{(s-4)(s-1)} + \frac{3}{(s-1)}$$

15

UNITEC
Universidad Tecnológica de México

③ Calcular la transformada inversa a $Y(s)$ para obtener $\boxed{y(t)}$

$$Y(s) = \underbrace{\frac{2}{(s-4)(s-1)}}_{\mathrm{I}} + \underbrace{\frac{3}{s-1}}_{\mathrm{II}}$$

I $F(s) = \frac{2}{s-4} \;\overset{\mathcal{L}^{-1}}{;}\; f(t) = 2e^{4t}$

$G(s) = \frac{1}{s-1} \;\overset{\mathcal{L}^{-1}}{;}\; g(t) = e^{t}$

II $\mathcal{L}^{-1}\left\{\frac{3}{s-1}\right\} = 3e^{t}$

I + II

$$y(t) = \frac{2}{3}e^{4t} - \frac{2}{3}e^{t} + 3e^{t}$$

$$\boxed{y(t) = \frac{2}{3}e^{4t} + \frac{7}{3}e^{t}}$$

$$f*g = \int_0^t (2e^{4\beta})(e^{t-\beta})\,d\beta = 2e^{t}\int_0^t e^{3\beta}\,d\beta = \frac{2}{3}e^{t}\left(e^{3\beta}\right)\Big|_0^t$$

$$f*g = \frac{2}{3}e^{t}\left[e^{3t} - 1\right] = \frac{2}{3}e^{4t} - \frac{2}{3}e^{t}$$

16

UNITEC
Universidad Tecnológica de México

EJERCICIO

1 $f(t) = e^{2t}$ $g(t) = \operatorname{senh}(t)$

a) $f * g$

b) $\mathcal{L}\{f * g\}$

2 $y' + 3y = t - 1$ $y(0) = 2$

17